

A sufficient condition and estimates of the frame bounds for generalized translation-invariant frames

Jakob Lemvig*, Jordy Timo van Velthoven†

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Abstract: We present a new sufficient condition for the frame property of generalized translation-invariant systems. The sufficient condition is formulated in the Fourier domain and includes estimates for the upper and lower frame bound. Contrary to previously known conditions of a similar nature, the estimates take the phase of the generating functions in consideration and not only their modulus. By considering the phase of the generating functions, these estimates allow phase cancellations to occur and lead to improvements of the known estimates. Moreover, the possibility of phase cancellations makes these estimates optimal for tight frames. Our results on generalized translation-invariant systems will be proved in the setting of locally compact abelian groups, but even in the euclidean setting and the special case of wavelet and shearlet systems our results are new and recover Tchamitchian’s estimate for dyadic wavelets.

1 Introduction

Deriving frame bound estimates for coherent frames has a long history in time-frequency and time-scale analysis. In this paper we consider sufficient conditions for the frame property of structured function systems that are based on properties of the generating functions in the Fourier domain. The first results of this nature go back to the very beginning of modern frame theory and the influential papers by Daubechies [13] and Daubechies, Grossmann and Meyer [15]. In [13], Daubechies provides general conditions on the generators and parameters of Gabor and wavelet systems to form a Bessel system or even a frame for $L^2(\mathbb{R})$. Moreover, these conditions provide estimates of the corresponding frame bounds of the considered function systems. These fundamental results attracted the attention of several groups of researchers [6, 11, 18, 28, 30, 31] and lead to improvements and generalizations of the results over the subsequent decade.

The best improvement of Daubechies’ frame bound estimates for Gabor systems was obtained by Ron and Shen [30]. This result asserts that if, for $P, Q \in \text{GL}_d(\mathbb{R})$ and $g \in L^2(\mathbb{R}^d)$,

$$B_0 := \text{ess sup}_{\omega \in (Q^T)^{-1}[0,1]^d} \frac{1}{|\det P|} \sum_{\alpha \in (P^T)^{-1}\mathbb{Z}^d} \left| \sum_{m \in Q\mathbb{Z}^d} \hat{g}(\omega - m) \overline{\hat{g}(\omega - m - \alpha)} \right| < \infty \quad (1.1)$$

*Corresponding author. Technical University of Denmark, Department of Applied Mathematics and Computer Science, Matematiktorvet 303B, 2800 Kgs. Lyngby, Denmark, E-mail: jakle@dtu.dk

†Technical University of Denmark, Department of Applied Mathematics and Computer Science, Matematiktorvet 303B, 2800 Kgs. Lyngby, Denmark, E-mail: s151426@student.dtu.dk

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and

$$A_0 := \operatorname{ess\,inf}_{\omega \in (Q^T)^{-1}[0,1]^d} \frac{1}{|\det P|} \left(\sum_{m \in Q\mathbb{Z}^d} |\hat{g}(\omega - m)|^2 - \sum_{\alpha \in (P^T)^{-1}\mathbb{Z}^d \setminus \{0\}} \left| \sum_{m \in Q\mathbb{Z}^d} \hat{g}(\omega - m) \overline{\hat{g}(\omega - m - \alpha)} \right| \right) > 0, \quad (1.2)$$

then the Gabor system $\{e^{2\pi i \langle m, \cdot \rangle} g(\cdot - n)\}_{m \in Q\mathbb{Z}^d, n \in P\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A_0 and B_0 . In fact, these sufficient conditions for Gabor systems are optimal for tight frames in the sense that the two frame bound estimates recover precisely the frame bound of a tight Gabor frame. This optimality stems from the fact that the provided frame bound estimates allow for phase cancellations over the modulation parameters $m \in Q\mathbb{Z}^d$.

For general wavelet systems in $L^2(\mathbb{R}^d)$, improved versions of Daubechies' frame bound estimates assert that if, for $A, C \in \operatorname{GL}_d(\mathbb{R})$, $B := A^T$ and $\psi \in L^2(\mathbb{R}^d)$,

$$b' := \operatorname{ess\,sup}_{\omega \in \mathbb{R}^d} \frac{1}{|\det C|} \sum_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} |\hat{\psi}(B^j \omega) \hat{\psi}(B^j \omega + (C^T)^{-1} k)| < \infty \quad (1.3)$$

and

$$a' := \operatorname{ess\,inf}_{\omega \in \mathbb{R}^d} \frac{1}{|\det C|} \left(\sum_{j \in \mathbb{Z}} |\hat{\psi}(B^j \omega)|^2 - \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}} |\hat{\psi}(B^j \omega) \hat{\psi}(B^j \omega + (C^T)^{-1} k)| \right) > 0, \quad (1.4)$$

then the wavelet system $\{|\det A|^{j/2} \psi(A^j \cdot - Ck)\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ forms a frame for $L^2(\mathbb{R}^d)$ with frame bounds a' and b' [6, 9, 28, 29]. In all these improved estimates the absolute sign is inside the sum over $j \in \mathbb{Z}$ due to which no cancellations over scales can occur. As a consequence, these estimates are not optimal for tight wavelet frames, and even for orthonormal bases the essential supremum in (1.3) might be positive infinity. In fact, no optimal sufficient conditions for general wavelet systems forming a tight frame are currently known. However, there is *one case* where the situation is known to improve, namely, for dyadic dilation $A = 2$ in dimension one. This result by Tchamitchian, as communicated by Daubechies [13, 14], states that a dyadic wavelet system $\{2^{j/2} \psi(2^j \cdot - ck)\}_{j \in \mathbb{Z}, k \in \mathbb{Z}, c > 0}$, in $L^2(\mathbb{R})$ satisfying

$$b_0 := \operatorname{ess\,sup}_{|\omega| \in [1,2]} \frac{1}{c} \sum_{q \in \mathbb{Z} \setminus 2\mathbb{Z}} \sum_{j \in \mathbb{Z}} \left| \sum_{n=0}^{\infty} \hat{\psi}(2^{j+n} \omega) \overline{\hat{\psi}(2^n(2^j \omega + q/c))} \right| < \infty \quad (1.5)$$

and

$$a_0 := \operatorname{ess\,inf}_{|\omega| \in [1,2]} \frac{1}{c} \left(\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \omega)|^2 - \sum_{q \in \mathbb{Z} \setminus 2\mathbb{Z}} \sum_{j \in \mathbb{Z}} \left| \sum_{n=0}^{\infty} \hat{\psi}(2^{j+n} \omega) \overline{\hat{\psi}(2^n(2^j \omega + q/c))} \right| \right) > 0 \quad (1.6)$$

forms a frame for $L^2(\mathbb{R})$ with frame bounds a_0 and b_0 . Tchamitchian's estimates¹ allow for cancellations over infinitely many scales $n \in \mathbb{N}_0$ and can be shown to be optimal for tight dyadic wavelet frames.

¹The estimates (1.5) and (1.6) are improved versions of the estimates that occur in [13, Theorem 2.9]. This improvement is similar to the way the estimates (1.3) and (1.4) improve on Daubechies' formulation [14, Proposition 3.3.2]. The improvement boils in essence down to a change of variable and taking suprema and infima differently than in the original proof.

The remarkable difference between the estimates for general wavelet systems and Gabor or dyadic wavelet systems lead in 2001 Christensen [8] to ask whether estimates that allow for cancellations can be obtained for non-dyadic wavelet systems. In 2007 Laugesen [27] answered the question negatively for transcendental dilations. For such dilations no cancellation over scales in frame bound estimates is possible. To the best knowledge of the authors no further results on the problem have appeared in the literature. However, in the light of Christensen's question and Laugesen's observation it is natural to ask which structured function systems allow for frame bound estimates with phase cancellation. This question has again become highly relevant with the arrival of new (directional) time-scale systems such as shearlets and curvelets. Indeed, for systems as shearlets, the only known method of constructing frames with compactly supported generators, which are used in applications, rely crucially on sufficient conditions similar to Daubechies' frame bound estimates for wavelets (1.3) and (1.4), but adapted to the shearlet setting [21, 24]. As a consequence, all known frame bound estimates for shearlet systems ignore phase cancellations.

In this work we are interested in sufficient conditions and frame bound estimates for a large class of structured function systems known as generalized translation-invariant (GTI) systems. In our main result, Theorem 2.3 in Section 2.2, we provide frame bound estimates with phase cancellations that are optimal for tight GTI frames. In $L^2(\mathbb{R}^d)$, a generalized translation-invariant system is a countable union of the form

$$\bigcup_{j \in J} \left\{ g_{j,p}(\cdot - \gamma) : \gamma \in C_j(\mathbb{R}^k \times \mathbb{Z}^{d-k}), p \in P_j \right\}, \quad (1.7)$$

where $\{C_j\}_{j \in J} \subset \text{GL}_d(\mathbb{R})$ and $\cup_{j \in J} \{g_{j,p}\}_{p \in P_j}$ is a given family of functions in $L^2(\mathbb{R}^d)$ indexed by P_j for $j \in J$. We will, however, work with GTI systems in the general setting of locally compact abelian groups, introduced in detail in Section 2.1, which allows us to consider structured function systems on \mathbb{Z}^d , $\mathbb{Z}/(N\mathbb{Z})$, \mathbb{R}^d , \mathbb{T}^d , or even vector valued functions, within the same formulation. Our general framework has the advantages that the results for Gabor systems, wave packet systems, wavelets, shearlets, curvelets, etc. will follow immediately as special cases; this holds even for their discrete, semi-continuous or continuous versions, which is obtained in (1.7) by modifying the parameter $0 \leq k \leq d$ and the index sets P_j . Indeed, Theorem 2.3 recovers the above mentioned sufficient conditions for Gabor systems and the Tchamitchian estimate for dyadic wavelet systems as special cases. The proof of Theorem 2.3 is based on the theory of almost periodic functions. Moreover, as we will see in Section 2.2, the general framework also leads to more transparent arguments than in the special cases.

In Section 2.3 we relate the estimates of Theorem 2.3 to previously known estimates for GTI systems. Section 3 is devoted to applications and examples of our main result. Gabor systems and wavelet systems in $L^2(\mathbb{R}^d)$ are considered in Section 3.1 and Section 3.2, respectively. Finally, we consider shearlets in $L^2(\mathbb{R}^2)$ in Section 3.3.

2 Sufficient conditions and frame bound estimates

The following subsection sets up notation and states results needed to prove our main result, which we will do in Section 2.2. In Section 2.3 we relate our main result to previously known estimates for generalized translation-invariant systems.

2.1 Generalized translation-invariant systems

Throughout this section, G will denote a second countable locally compact abelian group. The character group of G is denoted by \widehat{G} and forms a second countable locally compact abelian

group itself. The group operation in both G and \widehat{G} is written additively as $+$ and the identity element is denoted by 0 . The Haar measure on G will be denoted by μ_G . It is assumed that the Haar measure on G is given and that the Haar measure on \widehat{G} is the Plancherel measure. The subset $\Gamma \subseteq G$ will denote a closed, co-compact subgroup of G , i.e., the quotient space G/Γ is compact. In this case, the annihilator Γ^\perp of Γ is the countable, discrete subgroup $\Gamma^\perp := \{\omega \in \widehat{G} \mid \omega(x) = 0, \forall x \in \Gamma\}$. It is assumed that the Haar measure on Γ is given and that the Haar measure on G/Γ is the unique quotient measure provided by Weil's integral formula. Using this quotient measure $\mu_{G/\Gamma}$ on G/Γ , the density of the subgroup $\Gamma \subseteq G$ is defined as $d(\Gamma) := \mu_{G/\Gamma}(G/\Gamma)$.

The function systems defined next form the central object of this paper. Here, the translate of a function $f \in L^2(G)$ by $y \in G$ is denoted as $T_y f := f(\cdot - y)$.

Definition 2.1. Let J be a countable index set. For each $j \in J$, let $\Gamma_j \subseteq G$ be a closed, co-compact subgroup, and let P_j be an arbitrary (countable or uncountable) index set. For a given family of functions $\cup_{j \in J} \{g_{j,p}\}_{p \in P_j} \subset L^2(G)$, the collection of translates

$$\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$$

is called a *generalized translation-invariant (GTI) system* in $L^2(G)$.

Following [20], it is assumed that the generalized translation-invariant systems satisfy the following three *standing hypotheses*. For each $j \in J$:

- (I) The triple $(P_j, \Sigma_{P_j}, \mu_{P_j})$ forms a σ -finite measure space;
- (II) The mapping $(P_j, \Sigma_{P_j}) \rightarrow (L^2(G), \mathcal{B}_{L^2(G)})$, $p \mapsto g_{j,p}$ is Σ_{P_j} -measurable, where $\mathcal{B}_{L^2(G)}$ denotes the Borel σ -algebra on $L^2(G)$;
- (III) The mapping $(P_j \times G, \Sigma_{P_j} \otimes \mathcal{B}_G) \rightarrow (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$, $(p, x) \mapsto g_{j,p}(x)$ is $(\Sigma_{P_j} \otimes \mathcal{B}_G)$ -measurable, where \mathcal{B}_G denotes the Borel σ -algebra on G .

A generalized translation-invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is called a *generalized translation-invariant frame* for $L^2(G)$, with respect to $\{L^2(P_j \times \Gamma_j) \mid j \in J\}$, whenever there exist two constants $A, B > 0$, called the *frame bounds*, such that

$$A\|f\|^2 \leq \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) \leq B\|f\|^2 \quad (2.1)$$

for all $f \in L^2(G)$. A generalized translation-invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfying the upper frame bound is called a *Bessel system* or a *Bessel family* in $L^2(G)$. A frame for which the frame bounds can be chosen to be equal is called *tight*.

One of the consequences of the standing hypotheses is that the integrals in (2.1) are well-defined. We refer the reader to [20] for a detailed account on these assumptions and the frame theoretic properties of GTI systems.

In order to check whether a generalized translation-invariant system forms a Bessel system or a frame for $L^2(G)$, it suffices to check the frame condition on a dense subspace of $L^2(G)$. For a fixed Borel set $E \subset \widehat{G}$ satisfying $\mu_{\widehat{G}}(\overline{E}) = 0$, define the subset $\mathcal{D}_E(G)$ of $L^2(G)$ as

$$\mathcal{D}_E(G) = \{f \in L^2(G) \mid \hat{f} \in C_c(\widehat{G}), \text{supp } \hat{f} \cap E = \emptyset\},$$

where \hat{f} denotes the Fourier transform of $f \in L^2(G)$. The collection $\mathcal{D}_E(G)$ forms a dense subspace of $L^2(G)$. We consider the set E as fixed, but arbitrary. In applications, it usually suffices to take $E = \emptyset$ or $E = \{0\}$.

In the sequel, we need a regularity condition on the interaction of the subgroups Γ_j , the measure space P_j and the generators $g_{j,p}$. A GTI system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is said to satisfy the α -local integrability condition (α -LIC), with respect to E , if

$$\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\hat{G}} |\hat{f}(\omega) \hat{f}(\omega + \alpha) \hat{g}_{j,p}(\omega + \alpha) \hat{g}_{j,p}(\omega)| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty$$

for all $f \in \mathcal{D}_E(G)$.

Under a local integrability condition, the frame properties of GTI systems can be analyzed using the theory of almost periodic functions. This connection between almost periodic functions and GTI systems was discovered by Laugesen [25, 26] for wavelet systems and was extended to generalized shift-invariant (GSI) systems in $L^2(\mathbb{R}^d)$ by Hernández, Labate and Weiss [19]. A principal result in this connection between almost periodic functions and GTI systems is stated as the following result. Here, we use the formulation of this result based on the weakest local integrability condition, namely the α -LIC, by Jakobsen and Lemvig [20].

Theorem 2.2 ([19, 20, 22]). *Let $f \in \mathcal{D}_E(G)$. Suppose the system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the α -LIC. Then the function*

$$w_f : G \rightarrow \mathbb{C}, \quad w_f(x) := \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle T_x f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p)$$

is continuous, almost periodic and coincides point-wise with the generalized Fourier series

$$w_f(x) = \sum_{\alpha \in \cup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha(x),$$

where

$$c_\alpha = \int_{\hat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega + \alpha)} \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) d\mu_{\hat{G}}(\omega),$$

with $\kappa(\alpha) := \{j \in J \mid \alpha \in \Gamma_j^\perp\}$ for $\alpha \in \cup_{j \in J} \Gamma_j^\perp$.

2.2 The main result

This section contains the main result, Theorem 2.3, of this paper. Before proving this result, we mention the result [20, Proposition 3.7], which states that a generalized translation-invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfying

$$B' := \operatorname{ess\,sup}_{\omega \in \hat{G}} \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| d\mu_{P_j}(p) < \infty \quad (2.2)$$

and

$$A' := \operatorname{ess\,inf}_{\omega \in \hat{G}} \left(\sum_{j \in J} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) \right)$$

$$- \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp \setminus \{0\}} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| d\mu_{P_j}(p) \Big) > 0$$

forms a frame for $L^2(G)$ with lower bound A' and upper bound B' . The following result improves these frame bound estimates.

Theorem 2.3. *Let $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying the α -LIC.*

(i) *Suppose $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies*

$$B_0 := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \cup_{j \in J} \Gamma_j^\perp} \left| \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| < \infty, \quad (2.3)$$

then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forms a Bessel system in $L^2(G)$ with Bessel bound B_0 .

(ii) *Suppose $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies (2.3) and*

$$A_0 := \operatorname{ess\,inf}_{\omega \in \widehat{G}} \left(\sum_{j \in J} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) - \sum_{\alpha \in \cup_{j \in J} \Gamma_j^\perp \setminus \{0\}} \left| \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| \right) > 0,$$

then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forms a frame for $L^2(G)$ with lower bound A_0 and upper bound B_0 .

The proof of Theorem 2.3 is based on the simple estimate that a (generalized) Fourier series of an almost periodic function is bounded from above by the sum of the modulus of its coefficients and is bounded from below by the absolute value of its constant term minus the sum of the other terms in modulus.

Proof of Theorem 2.3. It suffices to show the frame inequalities on the dense subspace $\mathcal{D}_E(G)$ of $L^2(G)$.

Suppose the system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies condition (2.3) and the α -LIC with respect to E . For $\alpha \in \cup_{j \in J} \Gamma_j^\perp$, define the function $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$ by

$$t_\alpha(\omega) = \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p),$$

which is well-defined by condition (2.3). By Theorem 2.2, the non-negative function

$$w_f : G \rightarrow \mathbb{C}, \quad w_f(x) := \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle T_x f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p)$$

can be expressed as a generalized Fourier series $\sum_{\alpha \in \cup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha(x)$. Setting $x = 0$ gives

$$w_f(0) = \sum_{\alpha \in \cup_{j \in J} \Gamma_j^\perp} \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega + \alpha)} t_\alpha(\omega) d\mu_{\widehat{G}}(\omega)$$

$$= \int_{\widehat{G}} |\hat{f}(\omega)|^2 t_0(\omega) d\mu_{\widehat{G}}(\omega) + \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega + \alpha)} t_\alpha(\omega) d\mu_{\widehat{G}}(\omega). \quad (2.4)$$

Denote the series in the right-hand side expression above by R_f . An application of Beppo Levi's theorem and Young's inequality for products gives

$$\begin{aligned} |R_f| &\leq \int_{\widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |\hat{f}(\omega) \hat{f}(\omega + \alpha) t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\ &\leq \frac{1}{2} \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\ &\quad + \frac{1}{2} \int_{\widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |\hat{f}(\omega + \alpha)|^2 |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\ &= \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega), \end{aligned}$$

where the equality follows from the change of variable $\omega \mapsto \omega - \alpha$ and the identity

$$\overline{t_\alpha(\omega - \alpha)} = t_{-\alpha}(\omega).$$

Now it follows by (2.4) and the estimation of the term $|R_f|$ that, for all $f \in \mathcal{D}_E(G)$,

$$\sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) \leq B_0 \|f\|_2^2,$$

which shows (i). Assume now also that the assumption in (ii) is satisfied. Then

$$\sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) \geq \int_{\widehat{G}} |\hat{f}(\omega)|^2 t_0(\omega) d\mu_{\widehat{G}}(\omega) - |R_f| = A_0 \|f\|_2^2$$

as required. \square

Theorem 2.3 is best understood when it is rephrased in terms of the auxiliary functions $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$, $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, given by

$$t_\alpha(\omega) = \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p)$$

provided the series converges. In accordance with the tradition in Gabor theory, such functions will be called *auto-correlation functions*. Now, by considering the remainder function

$$R : \widehat{G} \rightarrow [0, \infty], \quad R(\omega) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)|,$$

the sufficient condition and frame bound estimates appearing in Theorem 2.3 simply read

$$B_0 = \operatorname{ess\,sup}_{\omega \in \widehat{G}} (t_0(\omega) + R(\omega)) = \operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)| < \infty \quad (2.5)$$

and

$$A_0 = \operatorname{ess\,inf}_{\omega \in \widehat{G}} (t_0(\omega) - R(\omega)) > 0. \quad (2.6)$$

Note that the infimum in (2.6) is well-defined once we assume (2.5).

The frame bound estimates of Theorem 2.3 are optimal for tight frames. That is, for a generalized translation-invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forming a tight frame, the estimates in Theorem 2.3 recover precisely the frame bound of the given frame. This simple observation is stated as the next result.

Proposition 2.4. *Let $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying the α -LIC. Suppose $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forms a tight frame for $L^2(G)$ with frame bound $A > 0$. Then $A = A_0 = B_0$.*

Proof. Suppose the system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a tight frame for $L^2(G)$ with bound $A > 0$. By [20, Theorem 3.4], it holds, for all $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, that

$$t_\alpha(\omega) = A\delta_{\alpha,0}$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$ and therefore $R(\omega) = 0$ $\mu_{\widehat{G}}$ -almost everywhere. The conclusion now follows. \square

2.3 The CC-condition and absolute CC-condition

In this section we compare the condition (2.3) appearing in Theorem 2.3 with the condition (2.2) and show that (2.3) is strictly weaker.

The condition (2.3) is referred to as the *CC-condition* for generalized translation-invariant systems. This condition forms a natural generalization of a well-known eponymous condition for Gabor systems as shown in Section 3.1. In [20], the term *absolute CC-condition* was coined for condition (2.2). We stress that the important difference between the CC-condition and the *absolute* CC-condition is the placement of the absolute sign in the summand. In the CC-condition, it is possible to have phase cancellations within each auto-correlation function while the absolute CC-condition prohibits such cancellations. Examples of wavelet systems for which such cancellations can occur and for which not are given in Section 3.2.

The following result shows that a generalized translation-invariant system satisfying the absolute CC-condition also satisfies the CC-condition.

Proposition 2.5. *Let $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system. Suppose $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the absolute CC-condition. Then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the CC-condition.*

Proof. Suppose the system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the absolute CC-condition. Then an application of Beppo Levi's theorem gives

$$\begin{aligned} \sum_{j \in J} \sum_{\alpha \in \Gamma_j^\perp} \left| \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| &\leq \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| d\mu_{P_j}(p) \\ &< \infty \end{aligned}$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$. Using the absolute convergence of the series, a re-ordering of the summation does not affect the convergence. Thus

$$\operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \left| \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| < \infty,$$

as required. \square

Jakobsen and the first named author [20] show that a generalized translation-invariant system satisfying the absolute CC-condition automatically satisfies the α -LIC. Thus the α -LIC is implicitly assumed in the estimate (2.2). The example by Bownik and Rzeszotnik [4, Example 3.2] with $N = 2$ shows that the CC-condition can hold, but that both the absolute CC-condition and the α -LIC fail. This might occur even for an orthonormal basis. However, any orthonormal basis satisfying the α -LIC also satisfies the CC-condition by Proposition 2.4. The Meyer wavelet is an example of an orthonormal basis for which the absolute CC-condition fails, but where the CC-condition and the α -LIC hold. Finally, we remark that Casazza, Christensen and Janssen [7] give an example of a Gabor system forming a Bessel system in $L^2(\mathbb{R})$, but where $\sum_{\alpha} |t_{\alpha}(\omega)| = \infty$ for a.e. $\omega \in \mathbb{R}$, which demonstrates that for Bessel systems both the CC-condition and the absolute CC-condition can fail even though the α -LIC holds.

3 Applications and examples

In this section the sufficient conditions given in Theorem 2.3 will be considered for special types of generalized translation-invariant systems. Throughout the section we consider multiple generators indexed by a not necessarily finite index set. In the examples, we will mostly be interested in explicit formulas for the auto-correlation functions t_{α} and the remainder function R as these are the ingredients of the frame bound estimates (2.5) and (2.6). Here, it should be understood that the (formal) expression for t_{α} might only be well-defined once we impose the CC-condition.

3.1 Gabor systems

Let L be a countable index set, let $\{g_{\ell}\}_{\ell \in L} \subset L^2(G)$, let $\Gamma \subseteq G$ be a closed, co-compact subgroup and let $\Lambda \subseteq \widehat{G}$ be such that equipping it with a σ -algebra Σ_{Λ} and a measure μ_{Λ} gives a measure space $(\Lambda, \Sigma_{\Lambda}, \mu_{\Lambda})$ satisfying the standard hypotheses. The (semi) co-compact Gabor system associated with the pair (Γ, Λ) is the collection of functions

$$\{E_{\lambda} T_{\gamma} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L} = \{\lambda(\cdot) g_{\ell}(\cdot - \gamma)\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L},$$

where $E_{\lambda} f(x) = \lambda(x) f(x)$ denotes the modulation operator on $L^2(G)$. The Gabor system $\{E_{\lambda} T_{\gamma} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ cannot be expressed as a generalized translation-invariant system, but it is unitarily equivalent to the translation-invariant system $\{T_{\gamma} E_{\lambda} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$. Therefore, the Gabor system $\{E_{\lambda} T_{\gamma} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ forms a Bessel system or a frame if, and only if, the corresponding translation-invariant system $\{T_{\gamma} E_{\lambda} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ forms a Bessel system or a frame. The system $\{T_{\gamma} E_{\lambda} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ forms a generalized translation-invariant system with $J = L$, $P = \Lambda$ and $g_{\lambda} = E_{\lambda} g$. The auto-correlation functions t_{α} associated with $\{T_{\gamma} E_{\lambda} g_{\ell}\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ can (formally) be written as

$$t_{\alpha}(\omega) = \sum_{\ell \in L} \int_{\Lambda} \hat{g}_{\ell}(\omega - \lambda) \overline{\hat{g}_{\ell}(\omega - \lambda - \alpha)} d\mu_{\Lambda}(\lambda)$$

for $\alpha \in \Gamma^\perp$. Any translation-invariant system satisfying the CC-condition also satisfies the α -LIC. Thus an application of Theorem 2.3 gives the frame bound estimates (2.5) and (2.6), where the 0th auto-correlation function t_0 is given by

$$t_0(\omega) = \sum_{\ell \in L} \int_{\Lambda} |\hat{g}_\ell(\omega - \lambda)|^2 d\mu_\Lambda(\lambda) \quad (3.1)$$

and the remainder function $R : \widehat{G} \rightarrow [0, \infty]$ by

$$R(\omega) = \sum_{\alpha \in \Gamma^\perp \setminus \{0\}} \left| \sum_{\ell \in L} \int_{\Lambda} \hat{g}_\ell(\omega - \lambda) \overline{\hat{g}_\ell(\omega - \lambda - \alpha)} d\mu_\Lambda(\lambda) \right|. \quad (3.2)$$

The frame bound estimates associated with (3.1) and (3.2) allow for phase cancellations over the modulation parameter $\lambda \in \Lambda$. Moreover, if Λ is a closed subgroup, we only need to take the essential supremum and infimum in (2.5) and (2.6), respectively, over a fundamental domain of Λ in \widehat{G} . For singly generated Gabor frames in $L^2(\mathbb{R}^d)$ associated with a pair of full-rank lattices (Λ, Γ) , the frame bound estimates (2.5) and (2.6) using (3.1) and (3.2) recover precisely the frame bound estimates (1.1) and (1.2) by Ron and Shen [30].

The sufficient conditions for Gabor frames are often formulated in the time domain. To do this, we switch the role of Γ and Λ and consider the Gabor system $\{E_\lambda T_\gamma g_\ell\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ as unitarily equivalent to the translation-invariant system $\{T_\lambda \mathcal{F}^{-1} T_\gamma g_\ell\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$, where \mathcal{F}^{-1} denotes the inverse Fourier transform. In this way, one obtains auto-correlation functions $s_\alpha : G \rightarrow \mathbb{C}, \alpha \in \Lambda^\perp$, given by

$$s_\alpha(x) := \sum_{\ell \in L} \int_{\Gamma} \overline{g_\ell(x - \gamma - \alpha)} g_\ell(x - \gamma) d\mu_\Gamma(\gamma),$$

provided the series converges. Hence, if

$$B_0 := \operatorname{ess\,sup}_{x \in G} \sum_{\alpha \in \Lambda^\perp} |s_\alpha(x)| < \infty$$

and

$$A_0 := \operatorname{ess\,inf}_{x \in G} \left(s_0(x) - \sum_{\alpha \in \Gamma^\perp \setminus \{0\}} |s_\alpha(x)| \right) > 0,$$

then $\{E_\lambda T_\gamma g_\ell\}_{\lambda \in \Lambda, \gamma \in \Gamma, \ell \in L}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A_0 and B_0 . For singly generated Gabor frames in $L^2(\mathbb{R}^d)$ associated with a pair of full-rank lattices (Λ, Γ) , these estimates recover precisely [16, Proposition 6.5.5].

3.2 Wavelet systems

Let J and L be countable index sets, let $\{\psi_\ell\}_{\ell \in L} \subset L^2(\mathbb{R}^d)$, let $A_j \in \operatorname{GL}_d(\mathbb{R})$ for $j \in J$, and let $\Gamma \subset \mathbb{R}^d$ be a full-rank lattice, i.e., $\Gamma = C\mathbb{Z}^d$ for some $C \in \operatorname{GL}_d(\mathbb{R})$. The wavelet system associated with the pair $(\{A_j\}_{j \in J}, \Gamma)$ is the collection of functions

$$\{D_{A_j} T_\gamma \psi_\ell\}_{j \in J, \gamma \in \Gamma, \ell \in L} = \{|\det A_j|^{1/2} \psi_\ell(A_j \cdot -\gamma)\}_{j \in J, \gamma \in \Gamma, \ell \in L}, \quad (3.3)$$

where $D_A f(x) = |\det A|^{1/2} f(Ax)$ is the unitary dilation operator on $L^2(\mathbb{R}^d)$. By considering the commutation relation $D_{A_j} T_\gamma = T_{A_j^{-1}\gamma} D_{A_j}$, the wavelet system can be considered as a

generalized translation-invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j}$ with $\Gamma_j = A_j^{-1}\Gamma$ and $g_{j,p} = D_{A_j}\psi_\ell$ for $j \in J$ and $p = \ell \in P$ with $P = L$ equipped with the counting measure.

Identify the annihilator Γ^\perp of the lattice Γ with its (dual) lattice $\Gamma^* = C^\sharp \mathbb{Z}^d$, where $C^\sharp := (C^T)^{-1}$. By considering the dual lattices $\Gamma_j^\perp = A_j^T \Gamma^* = A_j^T C^\sharp \mathbb{Z}^d$ and the corresponding index set $\kappa(\alpha) := \{j \in J \mid \alpha \in A_j^T \Gamma^*\}$, the auto-correlation functions $t_\alpha : \mathbb{R}^d \rightarrow \mathbb{C}$ can formally be written as

$$t_\alpha(\omega) = \frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \hat{\psi}_\ell(A_j^\sharp \omega) \overline{\hat{\psi}_\ell(A_j^\sharp(\omega + \alpha))} \quad (3.4)$$

for $\alpha \in \cup_{j \in J} A_j^T \Gamma^*$. Observe that $\kappa(0) = J$. Therefore, for wavelet systems satisfying the α -LIC, an application of Theorem 2.3 yields the frame bound estimates as in (2.5) and (2.6), where

$$t_0(\omega) = \frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in J} \left| \hat{\psi}_\ell(A_j^\sharp \omega) \right|^2$$

is the well-known Calderón sum, and the remainder function $R : \mathbb{R}^d \rightarrow [0, \infty]$ takes the form

$$R(\omega) = \frac{1}{|\det C|} \sum_{\alpha \in \cup_{j \in J} A_j^T \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \hat{\psi}_\ell(A_j^\sharp \omega) \overline{\hat{\psi}_\ell(A_j^\sharp(\omega + \alpha))} \right|. \quad (3.5)$$

Thus for all generators and for all scales in $\kappa(\alpha)$, we have the possibility of cancellations in the estimates for each $\alpha \in \cup_{j \in J} A_j^T \Gamma^* \setminus \{0\}$. This possibility of cancellations is in contrast to previously known sufficient conditions and frame bound estimates for wavelet systems; see, e.g., [9, Theorem 20.6.1]. These latter sufficient conditions uses the remainder function $\tilde{R} : \mathbb{R}^d \rightarrow [0, \infty]$ based on the absolute CC condition:

$$\tilde{R}(\omega) = \frac{1}{|\det C|} \sum_{j \in J} \sum_{\alpha \in A_j^T \Gamma^* \setminus \{0\}} \sum_{\ell \in L} \left| \hat{\psi}_\ell(A_j^\sharp \omega) \overline{\hat{\psi}_\ell(A_j^\sharp(\omega + \alpha))} \right|, \quad (3.6)$$

in which only the modulus of the generating functions are considered. To wrap up the discussion, we state the following result.

Theorem 3.1. *Let J, L be countable index sets, let $\{\psi_\ell\}_{\ell \in L} \subset L^2(\mathbb{R}^d)$, let $A_j \in \text{GL}_d(\mathbb{R})$ for $j \in J$ and let $\Gamma \subset \mathbb{R}^d$ be a full-rank lattice. Suppose the wavelet system $\{D_{A_j} T_\gamma \psi_\ell\}_{j \in J, \gamma \in \Gamma, \ell \in L}$ satisfies the α -LIC and satisfies*

$$b_0 := \text{ess sup}_{\omega \in \mathbb{R}^d} (t_0(\omega) + R(\omega)) < \infty \quad (3.7)$$

and

$$a_0 := \text{ess inf}_{\omega \in \mathbb{R}^d} (t_0(\omega) - R(\omega)) > 0, \quad (3.8)$$

where t_α and R are given in (3.4) and (3.5), respectively. Then $\{D_{A_j} T_\gamma \psi_\ell\}_{j \in J, \gamma \in \Gamma, \ell \in L}$ forms a frame for $L^2(\mathbb{R}^d)$ with bounds a_0 and b_0 .

The general form of the wavelet system in (3.3) allows us to handle wavelets of composite dilations, which we will do in the following example.

Example 1. Consider the Cartesian product $I \times J$ for two countable index sets I and J . Let $A_i, B_j \in \text{GL}_d(\mathbb{R})$ for $i \in I$ and $j \in J$. The wavelet system associated with the pair $(\{A_i B_j\}_{(i,j) \in I \times J}, \Gamma)$ is a collection of functions of the form

$$\{D_{A_i B_j} T_\gamma \psi_\ell\}_{i \in I, j \in J, \gamma \in \Gamma, \ell \in L}$$

and forms a so-called *wavelet system with composite dilations* in $L^2(\mathbb{R}^d)$, see e.g., [17]. One usually assumes that one of the two family of matrices, say $\{A_i\}_{i \in I}$, is volume preserving. We will assume that $A_i^T, i \in I$, acts invariant on Γ^* , that is, $A_i^T \Gamma^* = \Gamma^*$, e.g., in case $\Gamma = \mathbb{Z}^d$, this assumption reads $A_i \in \text{SL}_d(\mathbb{Z})$. Therefore, $\Gamma_{(i,j)}^\perp = B_j^T A_i^T \Gamma^* = B_j^T \Gamma^*$ for $(i, j) \in I \times J$. Thus, for composite wavelet systems satisfying the α -LIC, an application of Theorem 3.1 yields the frame bound estimates (3.7) and (3.8), where

$$t_0(\omega) = \frac{1}{|\det C|} \sum_{i \in I} \sum_{j \in J} \sum_{\ell \in L} \left| \hat{\psi}_\ell(A_i^\# B_j^\# \omega) \right|^2$$

and

$$R(\omega) = \frac{1}{|\det C|} \sum_{\alpha \in \bigcup_{j \in J} B_j^T \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \sum_{i \in I} \sum_{j \in \kappa(\alpha)} \hat{\psi}_\ell(A_i^\# B_j^\# \omega) \overline{\hat{\psi}_\ell(A_i^\# B_j^\# (\omega + \alpha))} \right|$$

with $\kappa(\alpha) := \{j \in J \mid \alpha \in B_j^T \Gamma^* \setminus \{0\}\}$.

The remainder of this subsection is devoted to examples for which phase cancellations in (3.5) can occur and for which such cancellations cannot be expected. In these examples a fixed dilation matrix is assumed. That is, it is assumed that $A_j = A^j$ for $j \in J = \mathbb{Z}$ for some $A \in \text{GL}_d(\mathbb{R})$. We first mention that the α -LIC for such wavelet systems is a very weak condition that can be ignored in most applications. For example, for a dilation matrix A being expansive, e.g., all eigenvalues of A are strictly greater than one in modulus, the wavelet system satisfies the α -LIC, with respect to $\{0\}$, precisely when the Calderón sum t_0 is in $L_{\text{loc}}^1(\mathbb{R}^d \setminus \{0\})$. See [2, Proposition 2.7]. More generally, for any pair (A^T, Γ) satisfying the *lattice counting estimate* introduced in [3], the α -LIC is satisfied whenever the CC-condition is. Bownik and the first named author [3] show that almost all wavelet systems satisfy the lattice counting estimate. The interested reader is referred to [3] for the precise statement.

In the following two examples the sufficient conditions for wavelet frames will be considered for two classes of (A, Γ) : one class, where the sets $\kappa(\alpha)$ are the smallest possible, and one class, where the sets $\kappa(\alpha)$ are large. For brevity, we will only be concerned with the particular form of the function $R(\omega)$ in (3.5) as the Calderon sum t_0 does not depend on the interaction of A and Γ . Moreover, we will in both examples assume that the α -LIC is satisfied.

Example 2. Let $A \in \text{GL}_d(\mathbb{R})$, let $B := A^T$ and let $\Gamma \subset \mathbb{R}^d$ be a full-rank lattice satisfying $\Gamma^* \cap B^j \Gamma^* = \{0\}$ for all $j \in \mathbb{Z} \setminus \{0\}$. Examples of such pairs (A, Γ) are $B = \beta I$ with I denoting the identity matrix, $\Gamma = \mathbb{Z}^d$, and $\beta \in \mathbb{R}$ being such that $\beta^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$. Now, since $B^j \Gamma^*, j \in \mathbb{Z}$, are disjoint outside the origin, it follows that the set $\kappa(\alpha)$ is a singleton for each $\alpha \in \bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}$. Therefore, the remainder function $R : \mathbb{R}^d \rightarrow [0, \infty]$ takes the form

$$\begin{aligned} R(\omega) &= \frac{1}{|\det C|} \sum_{\alpha \in \bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j} \omega) \overline{\hat{\psi}_\ell(B^{-j} (\omega + \alpha))} \right| \\ &= \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{k \in \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j} \omega) \overline{\hat{\psi}_\ell(B^{-j} \omega + k)} \right|. \end{aligned}$$

Consequently, phase cancellation between scales cannot occur in the estimates in Theorem 3.1. This observation fits precisely with a result by Laugesen [27]. In [27], it is proved that for wavelet systems in $L^2(\mathbb{R})$ with transcendental dilations $a > 0$ and integer translates, which in particular implies that $\bigcap_{j \in \mathbb{Z}} a^j \mathbb{Z} = \{0\}$, no cancellations between scales can happen for *any kind* of frame bound estimate. Note that despite the fact that no phase cancellations can happen, the estimate is still optimal for tight frames. This phenomenon is due to the fact that the characterizing equations for tight wavelet systems with expansive dilation A satisfying $\bigcap_{j \in \mathbb{Z}} B^j \Gamma^* = \{0\}$ are very restrictive on properties of ψ_ℓ . For example, Riesz bases possessing this property have to be combined MSF wavelets [1, 5, 12].

In the previous example it was assumed that the lattices $B^j \Gamma^*$, $j \in \mathbb{Z}$, are disjoint outside the origin. The next example assumes that the involved lattices are nested.

Example 3. Let $A \in \text{GL}_d(\mathbb{R})$, let $B := A^T$ and let $\Gamma \subset \mathbb{R}^d$ be a full-rank lattice satisfying $B\Gamma^* \subset \Gamma^*$. In case $\Gamma = \mathbb{Z}^d$, this assumption is equivalent with A being integer-valued. The union $\bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}$ can be re-written as the *disjoint* union $\bigcup_{m \in \mathbb{Z}} B^m(\Gamma^* \setminus B\Gamma^*)$. For $\alpha = B^m q$, where $m \in \mathbb{Z}$ and $q \in \Gamma^* \setminus B\Gamma^*$, we have that $\kappa(\alpha) = \{j \in \mathbb{Z} : j \leq m\}$. Therefore, the remainder function $R : \mathbb{R}^d \rightarrow [0, \infty]$ takes the form

$$\begin{aligned} R(\omega) &= \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus B\Gamma^*} \left| \sum_{j=-\infty}^m \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j}\omega) \overline{\hat{\psi}_\ell(B^{-j}(\omega + B^m q))} \right| \\ &= \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus B\Gamma^*} \left| \sum_{n=0}^{\infty} \sum_{\ell \in L} \hat{\psi}_\ell(B^{n+m}\omega) \overline{\hat{\psi}_\ell(B^n(B^m\omega + q))} \right|. \end{aligned} \quad (3.9)$$

Since the functions t_0 and R are B -dilation periodic, i.e., $t_0(B\omega) = t_0(\omega)$ and $R(B\omega) = R(\omega)$ for a.e. $\omega \in \mathbb{R}^d$, the estimates (3.7) and (3.8) read

$$b_0 = \operatorname{ess\,sup}_{\omega \in B(B(0,1)) \setminus B(0,1)} \left(\frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_\ell(B^j \omega) \right|^2 + R(\omega) \right)$$

and

$$a_0 = \operatorname{ess\,inf}_{\omega \in B(B(0,1)) \setminus B(0,1)} \left(\frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_\ell(B^j \omega) \right|^2 - R(\omega) \right),$$

where $B(0,1)$ denotes the unit ball in \mathbb{R}^d , and $R : \mathbb{R}^n \rightarrow [0, \infty]$ is given as in (3.9). For univariate wavelets with $A = B = 2$ and $\Gamma = c\mathbb{Z}$, $c > 0$, these estimates coincide with Tchamitchian's estimates (1.5) and (1.6).

To show that the frame bound estimates from Theorem 3.1 improve the sufficient condition based on the remainder function (3.6), note that (3.6) in the special case considered in this example simply reads

$$\tilde{R}(\omega) = \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{\alpha \in \Gamma^* \setminus \{0\}} \sum_{\ell \in L} \left| \hat{\psi}_\ell(B^j \omega) \overline{\hat{\psi}_\ell(B^j \omega + \alpha)} \right|.$$

Now to see that $R(\omega) \leq \tilde{R}(\omega)$ for a.e. $\omega \in \mathbb{R}^n$, one simply uses the triangle inequality and notes that there is a bijection between the indices $(m, n, q) \in (\mathbb{Z}, \mathbb{N}, \Gamma^* \setminus B\Gamma^*)$ and the indices $(j, \alpha) \in \mathbb{Z} \times \Gamma^* \setminus \{0\}$ given by

$$(m, n, q) \mapsto (j, \alpha), \quad \text{where } \alpha = B^n q \text{ and } j = n + m.$$

For simplicity of the discussion, let $\Gamma = \mathbb{Z}^d$. The above two examples show the two extremes on the possible phase cancellations of Theorem 2.3 that happen for integer dilations and certain irrational dilations. Laugesen [27] remarked that for rational dilation in dimension one, phase cancellations would be possible. In fact, for a rational dilation matrix $A \in \text{GL}_d(\mathbb{Q})$, phase cancellations over infinitely many scales are also possible, however, since the lattices $B^j\Gamma^*$ splits into families of nested lattices, the form of $\kappa(\alpha)$ becomes more complicated. For more information on wavelet frames with rational dilations, the interested reader is referred to [2, 10].

3.3 Shearlet systems

The classical shearlet system is a special case of wavelets with composite dilations. For simplicity we restrict our attention to $L^2(\mathbb{R}^2)$, but refer to [17, section 3.4] for a discussion of shearlet systems in $L^2(\mathbb{R}^d)$. One defines

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and considers the wavelet system associated with the pair $(\{S^k A^j\}_{j,k \in \mathbb{Z}}, \Gamma)$, where $\Gamma = C\mathbb{Z}^2$ for some $C \in \text{GL}_d(\mathbb{R})$. For the *classical shearlet system* $\{D_{S^k A^j} T_\gamma \psi_\ell\}_{j,k \in \mathbb{Z}, \gamma \in \Gamma, \ell \in L}$ we find as in Example 1 and Example 3 that the corresponding functions $t_0 : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $R : \mathbb{R}^2 \rightarrow [0, \infty]$ are formally given as

$$t_0(\omega) = \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{\ell \in L} \left| \hat{\psi}_\ell((S^\#)^k A^{-j} \omega) \right|^2 \quad (3.10)$$

and

$$R(\omega) = \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus A\Gamma^*} \left| \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} \sum_{\ell \in L} \hat{\psi}_\ell((S^\#)^k A^{n+m} \omega) \overline{\hat{\psi}_\ell((S^\#)^k A^n (A^m \omega + q))} \right|. \quad (3.11)$$

Since any shearlet system that satisfies the CC-condition satisfies the α -LIC, an application of Theorem 2.3 yields the following result.

Theorem 3.2. *Let L be a countable index set, let $\{\psi_\ell\}_{\ell \in L} \subset L^2(\mathbb{R}^2)$ and let $\Gamma \subset \mathbb{R}^2$ be a full-rank lattice. Suppose the shearlet system $\{D_{S^k A^j} T_\gamma \psi_\ell\}_{j,k \in \mathbb{Z}, \gamma \in \Gamma, \ell \in L}$ satisfies*

$$b_0 := \text{ess sup}_{\omega \in \mathbb{R}^2} (t_0(\omega) + R(\omega)) < \infty$$

and

$$a_0 := \text{ess inf}_{\omega \in \mathbb{R}^2} (t_0(\omega) - R(\omega)) > 0,$$

where t_0 and R are given in (3.10) and (3.11), respectively. Then $\{D_{S^k A^j} T_\gamma \psi_\ell\}_{j,k \in \mathbb{Z}, \gamma \in \Gamma, \ell \in L}$ forms a frame for $L^2(\mathbb{R}^2)$ with bounds a_0 and b_0 .

The estimates in Theorem 3.2 should be compared with previously used sufficient conditions for shearlet systems that are based on the absolute CC-condition and that do not allow for phase cancellations [23].

The rest of this section is devoted to cone-adapted shearlet systems. Such shearlets play a more important role in applications than the classical shearlets as they treat directions in an almost uniform manner. The cone-adapted shearlet system is a finite union of shift-invariant

systems and wavelet systems with composite dilations. To introduce these systems, we define $A_1 = A$, $S_1 = S$,

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For generators $\phi, \psi_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$, and full-rank lattices $\Gamma_i = C_i \mathbb{Z}^2$, $i = 0, 1, 2$, the *cone-adapted shearlet system* is given as:

$$\{T_\gamma \phi\}_{\gamma \in \Gamma_0} \cup \{D_{S_i^k A_i^j} T_\gamma \psi_i\}_{j \in \mathbb{N}_0, k \in \{-2^j, \dots, 2^j\}, \gamma \in \Gamma_i, i \in \{1, 2\}}.$$

For brevity we assume $\Gamma_i = \Gamma = C \mathbb{Z}^2$ for $i = 0, 1, 2$ for some $C \in \text{GL}_d(\mathbb{R})$. The auto-correlation functions $t_\alpha : \mathbb{R}^2 \rightarrow \mathbb{C}$, $\alpha \in \Gamma^*$, are then formally given as:

$$t_0(\omega) = |\hat{\phi}(\omega)|^2 + \sum_{i \in \{1, 2\}} \sum_{j=0}^{\infty} \sum_{k=-2^j}^{2^j} |\hat{\psi}_i((S_i^\sharp)^k A_i^{-j} \omega)|^2, \quad (3.12)$$

$$t_\alpha(\omega) = \hat{\phi}(\omega) \overline{\hat{\phi}(\omega + \alpha)} + \sum_{i \in \{1, 2\}} \sum_{j=0}^{m_i} \sum_{k=-2^j}^{2^j} \hat{\psi}_i((S_i^\sharp)^k A_i^{-j} \omega) \overline{\hat{\psi}_i((S_i^\sharp)^k A_i^{-j} (\omega + \alpha))}, \quad (3.13)$$

where $\alpha \in \Gamma^* \setminus \{0\}$, for each $i \in \{1, 2\}$, is written as $A_i^{m_i} q_i$ for unique $m_i \geq 0$ and $q_i \in \Gamma^* \setminus A_i \Gamma^*$. From the auto-correlation functions (3.13) we see that for the cases $\alpha \in C^\sharp \mathbb{Z}^2 \setminus 2C^\sharp \mathbb{Z}^2$ and $\alpha \in C^\sharp(4\mathbb{Z}^2 + (2, 2))$, the least amount of cancellation is possible. In this case the auto-correlation function reads

$$t_\alpha(\omega) = \hat{\phi}(\omega) \overline{\hat{\phi}(\omega + \alpha)} + \sum_{i \in \{1, 2\}} \sum_{k=-1}^1 \hat{\psi}_i((S_i^\sharp)^k \omega) \overline{\hat{\psi}_i((S_i^\sharp)^k (\omega + \alpha))},$$

hence only cancellation within the 0th scale is possible. On the other hand, when $\alpha \in 4^p C^\sharp \mathbb{Z}^2$ for some $p \in \mathbb{N}$, then cancellations can happen within all shears and all scales $j = 0, \dots, p$ for both shearlet generators ψ_1 and ψ_2 , that is, $m_1 = m_2 = p$ in (3.13).

As local integrability conditions can be ignored for shearlet systems, we arrive at the following Tchamitchian type estimate for cone-adapted shearlet systems.

Theorem 3.3. *Let $\phi, \psi_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$, and let Γ be a full-rank lattice in \mathbb{R}^2 . If*

$$b_0 := \text{ess sup}_{\omega \in \mathbb{R}^2} \sum_{\alpha \in \Gamma^*} |t_\alpha(\omega)| < \infty \quad (3.14)$$

and

$$a_0 := \text{ess inf}_{\omega \in \mathbb{R}^2} \left(t_0(\omega) - \sum_{\alpha \in \Gamma^* \setminus \{0\}} |t_\alpha(\omega)| \right) > 0, \quad (3.15)$$

where t_α is given by (3.12) and (3.13), then the cone-adapted shearlet system

$$\{T_\gamma \phi\}_{\gamma \in \Gamma} \cup \{D_{S_i^k A_i^j} T_\gamma \psi_i\}_{j \in \mathbb{N}_0, k \in \{-2^j, \dots, 2^j\}, \gamma \in \Gamma, i \in \{1, 2\}}.$$

is a frame for $L^2(\mathbb{R}^2)$ with bounds a_0 and b_0 .

The estimates in Theorem 3.3 are improvements of the sufficient conditions for cone-adapted shearlet systems as given in [21], which are based on the absolute CC-condition and do not allow for phase cancellations. Here, it should be noted that the conditions in [21] are currently the only known method for constructing cone-adapted shearlet frames with compactly supported generators. Moreover, the estimates without phase cancellation in [21] are used to “optimize” the choice of shearlet and translation lattice. It would be beneficial to instead use the improved estimates (3.14) and (3.15) for optimizing the construction of compactly supported shearlets.

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